Physics 319
Classical Mechanics

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Lecture 9

Hooke's Law

- For one dimensional simple harmonic motion force law is

$$F_x = -kx$$

- Corresponding potential function

$$U(x) = -\int_0^x (-kx)\,dx = \frac{kx^2}{2}$$

- Significance
  - Near an equilibrium $F_x = -dU/dx = 0$, so a quadratic approximation is the approximation to the potential with leading significance if $k \neq 0$ by Taylor's theorem
  - If $k > 0$, the motion exhibits, and is the quintessential example of, strong stability (motion under a perturbation stays near the motion without the perturbation)

Simple Harmonic Motion

- Energy Diagram

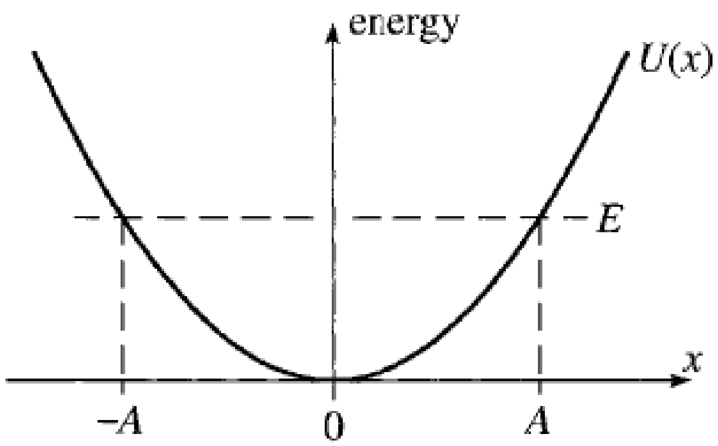


Figure 5.1 A mass m with potential energy $U(x) = \frac{1}{2}kx^2$ and total energy E oscillates between the two turning points at $x = \pm A$, where $U(x) = E$ and the kinetic energy is zero.

- The amplitude of the motion is denoted by A

Solutions for Simple Harmonic Motion

- Equation of motion

$$m\ddot{x} = -kx$$

$$\ddot{x} + \omega^2 x = 0$$

- $\omega = 2\pi f$ is the angular frequency of the oscillation
- Period of oscillation is $\tau = 2\pi / \omega$
- General solution

$$x(t) = B_c \cos\omega t + B_s \sin\omega t$$

- Equation is linear, and so superposition applies. Another way to write the general solution is

$$x(t) = C_+ e^{i\omega t} + C_- e^{-i\omega t}$$

$$\ddot{x}(t) = (i\omega)^2 C_+ e^{i\omega t} + (-i\omega)^2 C_- e^{-i\omega t}$$

$$= -(\omega)^2 C_+ e^{i\omega t} - (\omega)^2 C_- e^{-i\omega t} = -(\omega)^2 x(t)$$

- C_+ and C_- in general complex, and need $C_- = C_+^*$ for a *real* solution

Relation Between Expansion Coefficients

- Equating the two forms of the solution

$$B_c \cos \omega t + B_s \sin \omega t = B_c \frac{e^{i\omega t} + e^{-i\omega t}}{2} + B_s \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$C_+ = \left(B_c - iB_s\right)/2$$

$$C_- = \left(B_c + iB_s\right)/2$$

- Or going in the other direction

$$B_c = \left(C_+ + C_-\right)$$

$$B_s = \left(C_- - C_+\right)/i$$

- In other words

$$B_c = 2\,\mathrm{Re}\left(C_+\right) = 2\,\mathrm{Re}\left(C_-\right)$$

$$B_s = -2\,\mathrm{Im}\left(C_+\right)/i = +2\,\mathrm{Im}\left(C_-\right)/i$$

- Complex C_+ (C_-) allows one to handle the oscillation phase!

Solution in Amplitude-Phase Form

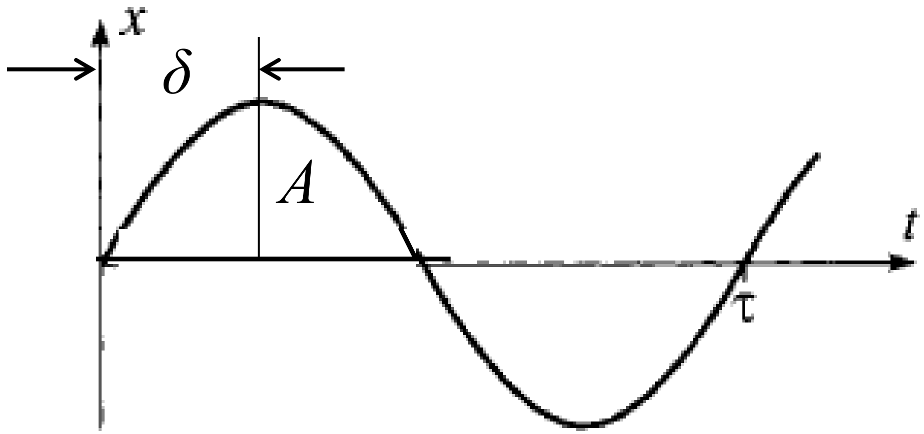
- Suppose we wish to write the general solution in amplitude-phase form

$$B_c \cos \omega t + B_s \sin \omega t = A \cos(\omega t - \delta)$$
$$= A \cos \omega t \cos \delta + A \sin \omega t \sin \delta$$
$$B_c^2 + B_s^2 = A^2 \left(\cos^2 \delta + \sin^2 \delta\right) = A^2$$
$$\tan \delta = B_s / B_c$$

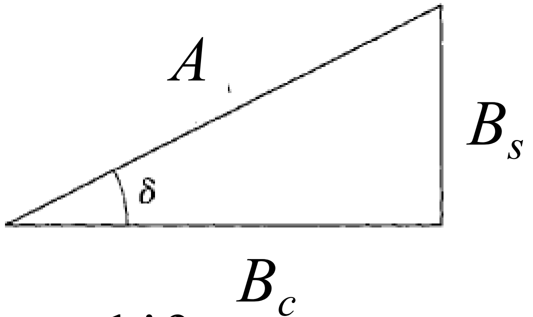
- Expression with complex representation even easier!

$$A^2 = (C_+ + C_-)^2 - (C_+ - C_-)^2 = 4C_+C_-$$
$$\delta = -\tan^{-1}\left(\operatorname{Im} C_+ / \operatorname{Re} C_+\right) = \tan^{-1}\left(\operatorname{Im} C_- / \operatorname{Re} C_-\right)$$
$$x(t) = 2\operatorname{Re}\left(C_+ e^{i\omega t}\right) = \operatorname{Re}\left[\left(B_c - iB_s\right)e^{i\omega t}\right]$$
$$= \operatorname{Re}\left[A e^{-i\delta} e^{i\omega t}\right]$$

Solution in Pictures



- Solution in amplitude phase form



- Computing the phase shift

Bottle in Bucket Example

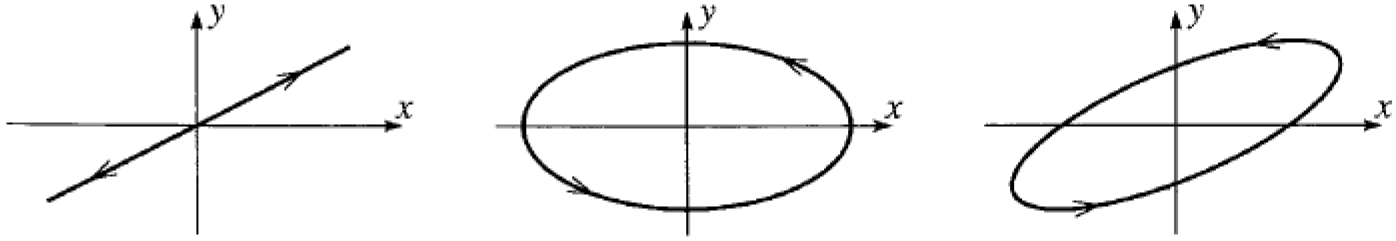
Energy Considerations

- Conserved total energy

$$
\begin{aligned}
E &= T + U \\
&= \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 \\
&= \frac{m}{2}\omega^2 A^2 \sin^2\left(\omega t - \delta\right) + \frac{k}{2}A^2\cos^2\left(\omega t - \delta\right) \\
&= \frac{k}{2}A^2
\end{aligned}
$$

2D Isotropic Oscillator

- Oscillations in two directions at same frequency

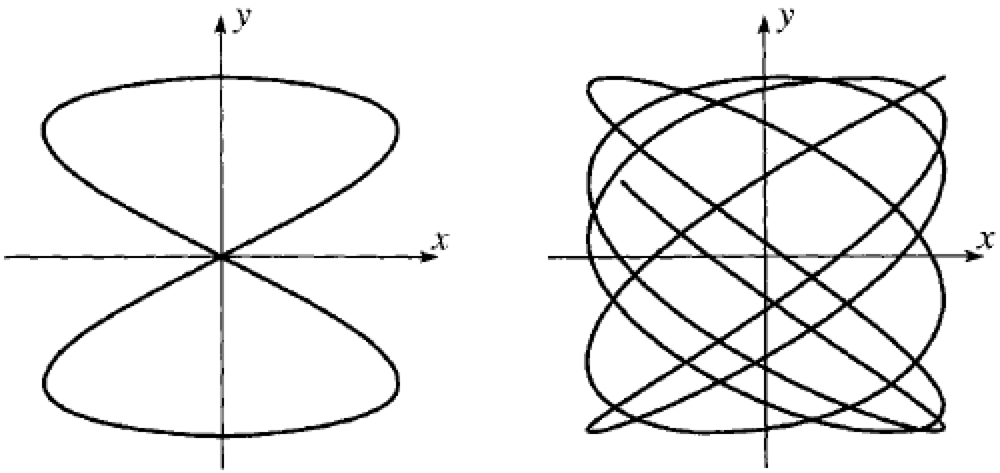


(a) $\delta = 0$ (b) $\delta = \pi/2$ (c) $\delta = \pi/4$

Figure 5.8 Motion of a two-dimensional isotropic oscillator as given by (5.20). **(a)** If $\delta = 0$, then x and y execute simple harmonic motion in step, and the point (x, y) moves back and forth along a slanting line as shown. **(b)** If $\delta = \pi/2$, then (x, y) moves around an ellipse with axes along the x and y axes. **(c)** In general (for example, $\delta = \pi/4$), the point (x, y) moves around a slanted ellipse as shown.

An-isotropic oscillations

- Lissajous figures when motion repeats itself (periodic)



(a) $\omega_x = 2\omega_y$ (b) $\omega_x = \sqrt{2}\,\omega_y$

Figure 5.9 **(a)** One possible path for an anisotropic oscillator with $\omega_x = 2$ and $\omega_y = 1$. You can see that x goes back and forth twice in the time that y does so once, and the motion then repeats itself exactly. **(b)** A path for the case $\omega_x = \sqrt{2}$ and $\omega_y = 1$ from $t = 0$ to $t = 24$. In this case the path never repeats itself, although, if we wait long enough, it will come arbitrarily close to any point in the rectangle bounded by $x = \pm A_x$ and $y = \pm A_y$.

Damped Motion

- Add a friction force (linear drag)

$$f_{frict} = -b\dot{x}$$

$$f_{tot} = -kx - b\dot{x}$$

$$m\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} \rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\beta = \frac{b}{2m}$$

- ω_0 is the frequency without damping
- Solution ansatz to linear ordinary differential equation (LCR circuit in electrical engineering)

$$x(t) = Ae^{i\omega t}$$

$$\left(-\omega^2 + 2\beta i\omega + \omega_0^2\right)Ae^{i\omega t} = 0$$

Homogeneous Solution

- For solutions to homogeneous equation

$$\left(-\omega^2+2\beta i\omega+\omega_0^2\right)=0$$

$$\omega=\beta i\pm\sqrt{\left(-4\beta^2+4\omega_0^2\right)/4}$$

$$x(t)=C_+e^{-\beta t}e^{i\sqrt{\omega_0^2-\beta^2}\,t}+C_-e^{-\beta t}e^{-i\sqrt{\omega_0^2-\beta^2}\,t}$$

- If $\beta=0$ (undamped) reduces to case before

$$x(t)=C_+e^{i\omega_0 t}+C_-e^{-i\omega_0 t}$$

- If $\beta \ll \omega_0$ (called the underdamped case), the square root is real, the angular frequency is adjusted to $\sqrt{\omega_0^2-\beta^2}$

$$x(t)=Ae^{-\beta t}\cos\left(\sqrt{\omega_0^2-\beta^2}\,t-\delta\right)$$

and the oscillation damps with exponential damping rate β

Over Damping and Critical Damping

- If $\beta >> \omega_0$ (called the overdamped case), the square root is imaginary, the damping has two rates and no oscillation

$$x(t) = C_+ e^{-\beta t} e^{\sqrt{\beta^2 - \omega_0^2}\, t} + C_- e^{-\beta t} e^{-\sqrt{\beta^2 - \omega_0^2}\, t}$$

- If $\beta = \omega_0$ (called the critically damped case), the square root vanishes. Need another method to determine second solution

$$\ddot{x} + 2\omega_0 \dot{x} + \omega_0^2 x = 0$$

$$x(t) = f(t) e^{-\omega_0 t}$$

$$\ddot{f} - 2\dot{f}\omega_0 + \omega_0^2 + 2\dot{f}\omega_0 - 2\omega_0^2 + \omega_0^2 = 0 \rightarrow \ddot{f} = 0$$

$$\therefore x(t) = Ce^{-\omega_0 t} + Dte^{-\omega_0 t}$$

- Motion dies out most quickly when the damping is critical

Solutions Qualitatively

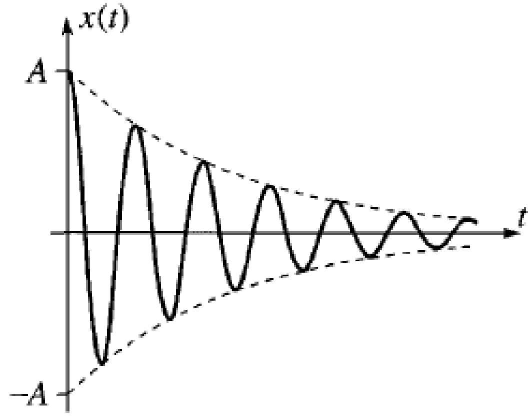


Figure 5.11 Underdamped oscillations can be thought of as simple harmonic oscillations with an exponentially decreasing amplitude $Ae^{-\beta t}$. The dashed curves are the envelopes, $\pm Ae^{-\beta t}$.

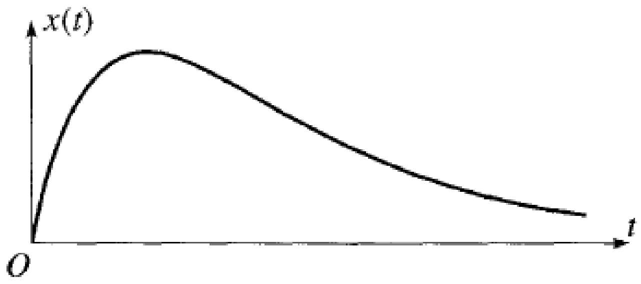


Figure 5.12 Overdamped motion in which the oscillator is kicked from the origin at $t = 0$. It moves out to a maximum displacement and then moves back toward O asymptotically as $t \to \infty$.